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1987 J. Phys. A: Math. Gen. 20 447

(http://iopscience.iop.org/0305-4470/20/2/030)

J. Phys. A: Math. Gen. 20 (1987) 447-454. Printed in the UK

Multicritical scaling in the magnetic hard-square lattice gas

Paul A Pearce

Mathematics Department, University of Melbourne, Parkville, Victoria 3052, Australia

Received 28 January 1986

Abstract. The scaling behaviour of the magnetic hard-square lattice gas is examined in the vicinity of the multicritical line located by the recent exact solution. A simple scaling form is proposed for the singular part of the free energy and corresponding scaling forms derived for the sublattice density difference, the magnetisation and the densities. These are all shown to be consistent with the exact results when restricted to the exact solution manifold. In the case of the densities this is achieved by proving appropriate elliptic function identities.

1. Introduction

The magnetic hard-square lattice gas is a three-state interaction-round-a-face or IRF model (Jimbo and Miwa 1985) that generalises and incorporates the magnetic Ising model and the hard-square (hexagon) lattice gas models. This general model has been solved exactly (Pearce 1985) on special two-dimensional manifolds in the full five-dimensional thermodynamic space spanned by the activity z, the diagonal (next-nearest-neighbour) lattice gas interactions L, M and the diagonal magnetic interactions J, K. These exact solution manifolds, denoted H (generalised hard hexagon), E (elliptic) and T (trigonometric), are given by

$$\text{H} \qquad \alpha = \beta = 0$$

$$z = (1 - e^{-L})(1 - e^{-M})/(e^{L+M} - e^{L} - e^{M}) \tag{1.1}$$

$$\text{E} \qquad e^{L} = (\alpha + \beta)/\beta(1 - \alpha^2)^{1/2} \qquad e^{M} = (\alpha + \beta)/\alpha(1 - \beta^2)^{1/2}$$

$$z = \alpha\beta(1 - \alpha^2)(1 - \beta^2)(1 + \alpha\beta)/(\alpha + \beta)^4 \tag{1.2}$$

$$\text{T} \qquad e^{L} = (1 - \alpha^2)^{1/2}/\beta^2 \qquad e^{M} = (1 - \beta^2)^{1/2}/\alpha^2$$

$$z = \alpha^2\beta^2 \tag{1.3}$$

where $\alpha = \tanh J$, $\beta = \tanh K$ and $z = z_+ + z_- = 2z_+$ is the total activity of the particles. For simplicity we will assume ferromagnetic interactions so that $\alpha, \beta \geqslant 0$.

The scaling behaviour of the non-magnetic hard-square (hexagon) models, with $J = K = 0$, has been examined by Huse (1983) on the basis of the exact solution (Baxter 1980, 1981, 1982, Baxter and Pearce 1982, 1983) on the H manifold. In particular, Huse conjectured simple scaling forms for the density functions which were subsequently confirmed (Pearce and Baxter 1984) on the H manifold using elliptic function identities. In this paper I examine the scaling behaviour in the vicinity of the line of multicritical points on the E manifold. The T manifold, which exhibits a line of essential singularities, will not be considered in this paper.

0305-4470/87/020447+08$02.50

2. Scaling and the E manifold

The two-dimensional E manifold (1.2) is divided into two distinct physical regimes EI and EII by a line of multicritical points located by the additional equation

$$\Delta_E \equiv (1-\alpha^2)(1-\beta^2)/\alpha\beta(1+\alpha\beta) = 1. \tag{2.1}$$

Regime EI ($\Delta_E > 1$) is a surface of twofold coexistence between the two paramagnetic ($\sqrt{2}\times\sqrt{2}$) square-ordered solid phases in which the particles preferentially occupy one of the two sublattices of next-nearest-neighbour sites. Regime EII ($\Delta_E < 1$) is a surface of fivefold coexistence between the four ferromagnetic ($\sqrt{2}\times\sqrt{2}$) square-ordered solid phases and the (paramagnetic) fluid phase.

The E manifold is more naturally parametrised with elliptic parameters u and q in place of α and β. The coordinate transformation relating u and q to α and β (or z, J, K, L and M), although complicated (equations (8a, b) of Pearce (1985)) is analytic both within regimes EI and EII and on the multicritical line. The parameter u appears in the arguments of the elliptic functions. It lies in the interval $0 \leq u \leq \pi/3$ and should be regarded as an 'anisotropy field', with $u = \pi/6$ corresponding to isotropic interactions ($J = K$, $L = M$). The parameter q is the nome of the elliptic functions and plays the role of a temperature-like variable or 'thermal scaling field'. In EI q lies in the interval $-1 < q < 0$ and in EII q lies in the interval $0 < q < 1$, $q = 0$ being the multicritical line.

The free energy per site can be written as

$$f = \ln \kappa = f_{\text{anal}} + f_{\text{sing}} \tag{2.2}$$

where f_{anal} and f_{sing} are the analytic and singular parts at the multicritical line. The known critical exponents (Pearce 1985) for the phase transition are

$$\begin{array}{llll} \text{EI} & \mu = \nu = \frac{3}{2} & \beta_1 = \frac{1}{8} & \\ \text{EII} & \mu = \nu = \frac{3}{2} & \beta = \frac{3}{16} & \beta_1 = \frac{1}{8} \end{array} \tag{2.3}$$

where the exponents μ, ν, β and β_1 refer to the interfacial tension, correlation length, magnetisation and sublattice density difference, respectively. Since the free energy varies analytically on E, even across the multicritical line, the exponent α is not properly defined. If, however, we assume the scaling relation

$$2 - \alpha = d\nu \tag{2.4}$$

for the specific heat exponent α, where $d = 2$ is the lattice dimension, we obtain

$$\alpha = -1. \tag{2.5}$$

These exponents appear to lie in a new universality class.

Using the above exponents and the functional form of the exact results (Pearce 1985) on the E manifold, the full scaling form of the singular part of the free energy near the multicritical line is expected to be

$$f_{\text{sing}} \approx q^3 \tilde{F}_{\pm}\left(\frac{\tilde{g}}{|g|^{5/2}}, \frac{\tilde{h}}{|q|^{45/16}}, \frac{\tilde{k}}{|q|^{23/8}}, u, a|q|^{3/2}, \ldots\right) \tag{2.6}$$

where $\tilde{g}$ is the 'leading thermal non-linear scaling field', $\tilde{h}$ is the 'magnetic symmetry breaking non-linear scaling field' and $\tilde{k}$ is the 'sublattice symmetry breaking non-linear scaling field'. The 'anisotropy field' u is a marginal operator, in renormalisation group language, that changes the scaling function but not the critical exponents. In (2.6) it is assumed that only one irrelevant scaling field $\tilde{g}_1$, with correction to scaling exponent $\frac{3}{2}$, contributes to the free energy, where $\tilde{g}_1 = a$ is constant and the most likely identification of a is the lattice spacing or spatial cutoff. The $\tilde{F}_+$ and $\tilde{F}_-$ functions are both analytic for $|q|<1$ and apply when $q\geqslant 0$ and $q\leqslant 0$ respectively. On the exact solution manifold E, $\tilde{g}=h=k=0$ and in this case, since the free energy is analytic on E (equation (9) in Pearce (1985)), we must have $F_\pm(u,x)=\tilde{F}_\pm(0,0,0,u,x)=0$.

The scaling forms for the sublattice density difference, magnetisation and densities can all be obtained by differentiating (2.6). Differentiating with respect to the sublattice symmetry breaking field k gives

$$\rho_1-\rho_2 \approx \left(\frac{\partial f_{\text{sing}}}{\partial k}\right)_{h=k=0} \approx \left(\frac{\partial \tilde{k}}{\partial k}\right) |q|^{1/8} \tilde{R}_\pm \left(\frac{\tilde{g}}{|q|^{5/2}}, u, a|q|^{3/2}, \ldots\right) \tag{2.7}$$

where $\tilde{R}_\pm$ is the derivative of $\tilde{F}_\pm$ with respect to its third argument. This is certainly consistent with the exact solution (Pearce 1985)

$$\begin{aligned} &\text{EI} \qquad \rho_1-\rho_2=(\sqrt{8}/3)|q|^{1/8}Q(q)Q(q^2)Q^2(q^3)Q^3(q^{12})/Q^7(q^6) \\ &\text{EII} \qquad \rho_1-\rho_2=\tfrac{4}{3}|q|^{1/8}Q(q)Q(q^2)Q(q^6)/Q^2(q^{3/2})Q(q^3) \end{aligned} \tag{2.8}$$

where

$$Q(x)=\prod_{n=1}^{\infty}(1-x^n). \tag{2.9}$$

Evidently, on the exact solution manifold ($\tilde{g}=h=k=0$), (2.7) can be written as the equality

$$\rho_1-\rho_2=\left(\frac{\partial \tilde{k}}{\partial k}\right) q^{1/8} R_{\text{I,II}}(q^{3/2}) \tag{2.10}$$

where $(\partial\tilde{k}/\partial k)$ and $R_{\text{I,II}}$ are analytic and, if we make the identification

$$\frac{\partial \tilde{k}}{\partial k}(q)=Q(q)Q(q^2) \tag{2.11}$$

the scaling functions are given by

$$\begin{aligned} R_{\text{I}}(x)&=(\sqrt{8}/3)Q^2(x^2)Q^3(x^8)/Q^7(x^4) \\ R_{\text{II}}(x)&=\tfrac{4}{3}Q(x^4)/Q^2(x)Q(x^2). \end{aligned} \tag{2.12}$$

Similarly, differentiating (2.6) with respect to the magnetic field h gives

$$m \approx \left(\frac{f_{\text{sing}}}{\partial h}\right)_{h=k=0} \approx \left(\frac{\partial \tilde{h}}{\partial h}\right) |q|^{3/16} \tilde{M}_\pm \left(\frac{\tilde{g}}{|q|^{5/2}}, u, a|q|^{3/2}, \ldots\right) \tag{2.13}$$

where $\tilde{M}_\pm$ is the derivative of $\tilde{F}_\pm$ with respect to its second argument. Again, this is consistent with the exact results (Pearce 1985)

$$\begin{aligned} &\text{EI} \qquad m=\tfrac{1}{2}(m_1+m_2)=0 \\ &\text{EII} \qquad m=\sqrt{\tfrac{2}{3}}\,q^{3/16}\frac{Q^2(q)}{Q(q^2)}\frac{Q(q^3)Q(q^6)}{Q^3(q^{3/2})} \end{aligned} \tag{2.14}$$

and on the exact solution manifold ($\tilde{g}=h=k=0$) we can write the magnetisation as

$$m=\left(\frac{\partial \tilde{h}}{\partial h}\right) q^{3/16} M_{\text{I,II}}(q^{3/2}) \tag{2.15a}$$

with

$$\frac{\partial \tilde{h}}{\partial h}(q) = Q^2(q)/Q(q^2)$$

$$M_{\mathrm{I}}(x) = 0 \tag{2.15b}$$

$$M_{\mathrm{II}}(x) = \sqrt{\tfrac{2}{3}}\,Q(x^2)Q(x^4)/Q^3(x).$$

If we assume that only those scaling fields exhibited in (2.6) contribute, we find that the singular part of the density is given by

$$\rho_{\mathrm{sing}} = z\left(\frac{\partial f_{\mathrm{sing}}}{\partial z}\right) = z\left(\frac{\partial q}{\partial z}\right)\left(\frac{\partial f_{\mathrm{sing}}}{\partial q}\right) + z\left(\frac{\partial \tilde{g}}{\partial z}\right)\left(\frac{\partial f_{\mathrm{sing}}}{\partial \tilde{g}}\right). \tag{2.16}$$

On the E manifold, however, $(\partial f_{\mathrm{sing}}/\partial q)$ vanishes identically so the scaling form of the densities should be

$$\rho_{\mathrm{sing}} = q^{1/2}X(q)\,Y_{\mathrm{I,II}}(q^{3/2}) \tag{2.17}$$

where the X and Y functions are analytic. The exact results for the densities on the E manifold are (Pearce 1985)

$$\text{EI} \qquad \rho_{\mathrm{I}}^{\mathrm{solid}} = \tfrac{1}{2}(\rho_1+\rho_2) = D(q) \equiv \frac{1}{3}\,\frac{Q^2(q)Q(q^2)}{Q(q^4)}\,\frac{Q^2(q^3)Q^3(q^{12})}{Q^7(q^6)}$$

$$\text{EII} \qquad \rho_{\mathrm{II}}^{\mathrm{fluid}} = D(q^{1/2}) \tag{2.18}$$

$$\rho_{\mathrm{II}}^{\mathrm{solid}} = \tfrac{1}{2}(\rho_1+\rho_2) = D(-q^{1/2})$$

where the critical (multicritical) density is $\rho_c = D(0) = \frac{1}{3}$. It is not at all obvious from these expressions that the singular parts of the densities (2.18) have the required scaling form (2.17).

One definition of the singular part of the density is the difference between its value for $q>0$ and the value obtained by analytically continuing from $q<0$ around the singularity at $q=0$. Alternatively, since the densities (2.18) admit Taylor expansions in powers of $q^{1/2}$, we can take $\rho_{\mathrm{sing}} = \frac{1}{2}[\rho(q^{1/2}) - \rho(-q^{1/2})]$ to be the odd part of $\rho(q^{1/2})$. In appendices 1 and 2 we will prove the identities

$$D(x) - D(-x) = -\tfrac{4}{3}xQ(x^2)Q(x^4)Q^5(x^{12})/Q^7(x^6) \tag{2.19a}$$

$$D(x) - D(x^2) = -\tfrac{2}{3}xQ(x^2)Q(x^4)Q^4(x^3)Q^4(x^{24})/Q^5(x^6)Q^5(x^{12}). \tag{2.19b}$$

From these and (2.18) it follows immediately that

$$\rho_{\mathrm{II}}^{\mathrm{fluid}} - \rho_{\mathrm{I}}^{\mathrm{solid}} = D(q^{1/2}) - D(q) = q^{1/2}Q(q)Q(q^2)\,Y_{\mathrm{I,II}}^{\mathrm{fluid}}(q^{3/2})$$

$$\rho_{\mathrm{II}}^{\mathrm{solid}} - \rho_{\mathrm{I}}^{\mathrm{solid}} = D(-q^{1/2}) - D(q) = q^{1/2}Q(q)Q(q^2)\,Y_{\mathrm{I,II}}^{\mathrm{solid}}(q^{3/2})$$

$$(\rho_{\mathrm{I}})_{\mathrm{sing}} = 0 = q^{1/2}Q(q)Q(q^2)\,Y_{\mathrm{I}}^{\mathrm{solid}}(q^{3/2}) \tag{2.20a}$$

$$(\rho_{\mathrm{II}}^{\mathrm{fluid}})_{\mathrm{sing}} = \tfrac{1}{2}[D(q^{1/2}) - D(-q^{1/2})] = q^{1/2}Q(q)Q(q^2)\,Y_{\mathrm{II}}^{\mathrm{fluid}}(q^{3/2})$$

$$(\rho_{\mathrm{II}}^{\mathrm{solid}})_{\mathrm{sing}} = \tfrac{1}{2}[D(-q^{1/2}) - D(q^{1/2})] = q^{1/2}Q(q)Q(q^2)\,Y_{\mathrm{II}}^{\mathrm{solid}}(q^{3/2})$$

where

$$Y_{\mathrm{I,II}}^{\mathrm{fluid}}(x) = -Y_{\mathrm{I,II}}^{\mathrm{solid}}(-x) = -\tfrac{2}{3}Q^4(x)Q^4(x^8)/Q^5(x^2)Q^5(x^4)$$

$$Y_{\mathrm{I}}^{\mathrm{solid}}(x) = 0 \tag{2.20b}$$

$$Y_{\mathrm{II}}^{\mathrm{fluid}}(x) = -Y_{\mathrm{II}}^{\mathrm{solid}}(-x) = -\tfrac{2}{3}Q^5(x^4)/Q^7(x^2).$$

These results verify the simple scaling form (2.17) in detail and for both definitions of $\rho_{\rm sing}$ with

$$X(q)=Q(q)Q(q^2). \tag{2.21}$$

Before proceeding to prove the two identities (2.19), we observe that the correlation length should scale with the same variables as the free energy so, since $\nu=\frac{3}{2}$, we expect

$$\xi^{-1}\approx|q|^{3/2}\tilde{\Xi}_\pm\left(\frac{\tilde{g}}{|q|^{5/2}}, u, a|q|^{3/2},\ldots\right). \tag{2.22}$$

Again this is consistent with the exact results, with in fact no dependence on u, and may be written as the equality

$$\xi^{-1}=|q|^{3/2}\Xi_\pm(|q|^{3/2}). \tag{2.23}$$

Actually, there are two correlation lengths ξ_m and ξ_ρ corresponding to the decay of magnetisation-magnetisation correlations and density-density correlations respectively. The exact results (Pearce 1985) are

$$\begin{aligned} &\text{EI} \qquad 2\xi_m^{-1}=\xi_\rho^{-1}=-\ln k'(|q|^{3/2})\\ &\text{EII} \qquad \xi_m^{-1}=\xi_\rho^{-1}=-\ln k'(|q|^{3/2}) \end{aligned} \tag{2.24}$$

where

$$k'(x)=\prod_{n=1}^{\infty}[(1-x^{2n-1})/(1+x^{2n-1})]^4. \tag{2.25}$$

Appendix 1. The first identity

The identities (2.19) were first obtained by computer calculations as in Pearce and Baxter (1984). To prove these identities analytically we will need the elliptic functions ($|q|<1$)

$$\begin{aligned} f(w,q)&=\prod_{n=1}^{\infty}(1-q^{n-1}w)(1-q^nw^{-1})(1-q^n)\\ &=\sum_{n=-\infty}^{\infty}(-1)^nq^{n(n-1)/2}w^n=f(qw^{-1},q) \end{aligned} \tag{A1.1}$$

$$\begin{aligned} \theta_4(u,q)&=\prod_{n=1}^{\infty}(1-2q^{2n-1}\cos 2u+q^{4n-2})(1-q^{2n})\\ &=\sum_{n=-\infty}^{\infty}(-1)^nq^{n^2}\,e^{2nui}=f(q\,e^{2ui},q^2). \end{aligned} \tag{A1.2}$$

Using the simple identities

$$\begin{aligned} \theta_4(0,q)&=Q^2(q)/Q(q^2)=f(q,q^2)\\ \theta_4(\pi/2,q)&=Q^5(q^2)/Q^2(q)Q^2(q^4)=f(-q,q^2) \end{aligned} \tag{A1.3}$$

we see that

$$\begin{aligned} D(x)&=\frac{1}{3}\frac{Q^2(x)Q(x^2)}{Q(x^4)}\frac{Q^2(x^3)Q^3(x^{12})}{Q^7(x^6)}\\ &=\frac{1}{3}\frac{\theta_4(0,x)\theta_4(0,x^2)}{\theta_4(\pi/2,x^3)\theta_4(0,x^6)} \end{aligned} \tag{A1.4}$$

and hence the first identity in (2.19) becomes

$$D(x)-D(-x)=\frac{\theta_4(0,x^2)}{3\theta_4(0,x^6)\theta_4(0,x^3)\theta_4(\pi/2,x^3)}$$
$$\times[\theta_4(0,x)\theta_4(0,x^3)-\theta_4(\pi/2,x)\theta_4(\pi/2,x^3)]. \tag{A1.5}$$

But now let $z=\exp(2\pi i/4)$ be a fourth root of unity. Then, using the fact that $\theta_4(\pi/4,x)=\theta_4(3\pi/4,x)$ and the series representation (A1.2), we obtain

$$\theta_4(0,x)\theta_4(0,x^3)-\theta_4(\pi/2,x)\theta_4(\pi/2,x^3)$$
$$=\sum_{r=0}^{3} z^r\theta_4(\pi r/4,x)\theta_4(\pi r/4,x^3)$$
$$=\sum_{r=0}^{3}\sum_{m,n=-\infty}^{\infty}(-1)^{m+n}x^{m^2+3n^2}\exp[(m+n+1)2\pi i r/4]$$
$$=-4\sum_{n,k=-\infty}^{\infty}x^{12k^2-8nk+4n^2+2n-8k+1} \tag{A1.6}$$
$$=-4\sum_{k,l=-\infty}^{\infty}x^{12k^2+4l^2-6k+2l+1}$$
$$=-4xf(-x^2,x^8)f(-x^6,x^{24}).$$

Here the sum on m has been restricted to the values $m=4k-n-1$ because of the relation

$$\tfrac{1}{4}\sum_{r=0}^{3} z^{pr}=\begin{cases}1 & p=0\ (\mathrm{mod}\ 4)\\ 0 & p\neq 0\ (\mathrm{mod}\ 4)\end{cases}. \tag{A1.7}$$

Next, the double series was factored by transforming from n to $l=n-k$. Finally, putting (A1.6) into (A1.5) and using the simple identities (A1.3) and

$$f(-x,x^4)=Q^2(x^2)/Q(x) \tag{A1.8}$$

we obtain

$$D(x)-D(-x)=-\frac{4}{3}x\frac{Q^2(x^2)Q^3(x^{12})}{Q(x^4)Q^6(x^6)}\frac{Q^2(x^4)Q^2(x^{12})}{Q(x^2)Q(x^6)}$$
$$=-\tfrac{4}{3}xQ(x^2)Q(x^4)Q^5(x^{12})/Q^7(x^6) \tag{A1.9}$$

which is (2.19*a*).

Appendix 2. The second identity

To prove the second identity (2.19*b*) we will first prove the auxiliary identity

$$3D(x)\equiv\frac{Q^2(x)Q(x^2)}{Q(x^4)}\frac{Q^2(x^3)Q^3(x^{12})}{Q^7(x^6)}=1-2x\frac{f(-x,x^6)f(x^2,x^{12})}{f(-x^3,x^6)f(x^6,x^{12})}. \tag{A2.1}$$

Starting with (A1.4) and using the identity $\theta_4(0,x)=\theta_4(\pi/4,x^{1/4})$ we find

$$3D(x)-1=\frac{\theta_4(0,x^2)\theta_4(\pi/4,x^{1/4})-\theta_4(\pi/2,x^3)\theta_4(0,x^6)}{\theta_4(\pi/2,x^3)\theta_4(0,x^6)} \tag{A2.2}$$

From (A1.3) the denominator here is

$$\theta_4(\pi/2, x^3)\theta_4(0, x^6) = f(-x^3, x^6)f(x^6, x^{12}) \tag{A2.3}$$

as in (A2.1). It therefore remains to show that the numerator in (A2.2) is given by

$$\theta_4(0, x^2)\theta_4(\pi/4, x^{1/4}) - \theta_4(\pi/2, x^3)\theta_4(0, x^6) = -2xf(-x, x^6)f(x^2, x^{12}). \tag{A2.4}$$

This identity follows immediately from two further identities:

$$\theta_4(0, x^2)\theta_4(3\pi/12, x^{1/4}) + \theta_4(\pi/3, x^2)\theta_4(7\pi/12, x^{1/4}) + \theta_4(2\pi/3, x^2)\theta_4(11\pi/12, x^{1/4})$$
$$= 3\theta_4(\pi/2, x^3)\theta_4(0, x^6) \tag{A2.5a}$$

$$\theta_4(0, x^2)\theta_4(3\pi/12, x^{1/4}) - \tfrac{1}{2}\theta_4(\pi/3, x^2)\theta_4(7\pi/12, x^{1/4}) - \tfrac{1}{2}\theta_4(2\pi/3, x^2)\theta_4(11\pi/12, x^{1/4})$$
$$= -3xf(-x, x^6)f(x^2, x^{12}). \tag{A2.5b}$$

The two identities (A2.5) can be proved analogously to (A1.6). Let $z = \exp(2\pi i/6)$ be a sixth root of unity so that

$$\tfrac{1}{6}\sum_{r=0}^{5} z^{pr} = \begin{cases} 1 & p = 0 \pmod 6 \\ 0 & p \neq 0 \pmod 6 \end{cases}. \tag{A2.6}$$

Then the identities (A2.5) are obtained by manipulating a double series representation as follows:

$$6^{-1}\sum_{r=0}^{5} z^{2pr}\theta_4(\pi r/3, x^2)\theta_4(\pi/4 + \pi r/6, x^{1/4})$$
$$= \tfrac{1}{6}\sum_{r=0}^{5}\sum_{m,n=-\infty}^{\infty}(-1)^{m+n}x^{2m^2+n^2/4}\,e^{n\pi i/2}z^{(2m+n+2p)r}$$
$$= \sum_{m,k=-\infty}^{\infty}(-1)^{k+p}x^{2m^2+(3k-m-p)^2} \tag{A2.7}$$
$$= \sum_{l,k=-\infty}^{\infty}(-1)^{k+p}x^{3l^2+6k^2+2p(l-2k)+p^2}$$
$$= (-1)^p x^{p^2} f(-x^{3-2p}, x^6)f(x^{6-4p}, x^{12}).$$

Here the sum on n has been restricted to the values $n = 6k - 2m - 2p$ using (A2.6) and the resulting double series was factored by transforming from m to $l = m - k$. The particular identities (A2.5a) and (A2.5b) are obtained by choosing $p = 0$ and $p = 1$, respectively, in (A2.7), and using the simple facts that

$$\theta_4(u, q) = \theta_4(\pi + u, q) = \theta_4(\pi - u, q) \tag{A2.8}$$

$$z^2 + z^4 = 2\cos(2\pi/3) = -1. \tag{A2.9}$$

This concludes the proof of (A2.1), which using the simple identity

$$f(-w, q)/f(q, q^2) = f(w^2, q^2)/f(w, q) \tag{A2.10}$$

can now be written as

$$3D(x) - 1 = -2x\frac{f(x^3, x^6)}{f(x, x^6)}\frac{f^2(x^2, x^{12})}{f^2(x^6, x^{12})}$$
$$= -2x\frac{f^2(x^2, x^{12})f^2(x^3, x^{12})}{f(x, x^{12})f(x^5, x^{12})f^2(x^6, x^{12})}. \tag{A2.11}$$

Using this last form for $D(x)$, the identity (2.19*b*) to be proved becomes

$$\frac{f^2(x^2,x^{12})f^2(x^3,x^{12})}{f(x,x^{12})f(x^5,x^{12})f^2(x^6,x^{12})}-x\frac{f^2(x^4,x^{24})f^2(x^6,x^{24})}{f(x^2,x^{24})f(x^{10},x^{24})f^2(x^{12},x^{24})}$$

$$=Q(x^2)Q(x^4)\frac{Q^4(x^3)Q^4(x^{24})}{Q^5(x^6)Q^5(x^{12})}. \qquad (A2.12)$$

Using (A2.10) this finally simplifies to

$$f(-x,x^{12})f(-x^5,x^{12})f^2(-x^6,x^{12})-xf^2(-x^2,x^{12})f^2(-x^3,x^{12})$$

$$=Q(x^2)Q(x^4)\frac{Q^4(x^3)Q^4(x^{24})}{Q^5(x^6)Q^5(x^{12})}\frac{f(x^2,x^{24})f(x^{10},x^{24})f^6(x^{12},x^{24})}{f^2(x^2,x^{12})f^2(x^3,x^{12})}$$

$$\equiv f^2(x^4,x^{12})f^2(x^3,x^{12}). \qquad (A2.13)$$

But this is just a special case ($a=x^4$, $b=-x^6$, $c=-x^3$, $d=1$, $q=x^{12}$) of the very general identity

$$f(ac)f(a/c)f(bd)f(b/d)-(b/c)f(ab)f(a/b)f(cd)f(c/d)=f(ad)f(a/d)f(bc)f(b/c) \qquad (A2.14)$$

where a, b, c, d are complex and $f(w)\equiv f(w,q)$. To prove this general identity let $F(a)$ be the ratio of the LHS of (A2.14) over the RHS. Then $F(a)=F(qa)$ is analytic throughout a period annulus and hence is constant by Liouville's theorem. Setting $a=c$ verifies that the constant is unity. This completes the proof of the second scaling identity (2.19*b*).

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